Entropy rates of non-deterministic cellular automata

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Understanding how information flows through spatially distributed media.

Studying the “conformist” grid: how do people make decisions?

**Basic Setup**

- Cellular automata (CA), on a grid, in discrete time
- Non-deterministic state machines
- At each time, each node observes its neighbors, and draws from a Bernoulli distribution, $B(p)$.
- $0 < p < 1$, $p$ increases with the number of neighbors of value $1$, and is symmetric (if all values are flipped, $p' = 1 - p$).
Notation

- $X, X_0$: value of given node at given time
- $X^n$: collection of neighbors of node $X$
- $X_i \ (i > 0)$: the value of neighbor of $X$
- $X_n$: the number of neighbors of $X$ with value 1
- $X^{-i}$: collection of values excluding value $i$
- $Y$: value of given node at next time
**Markov Chains** Determine steady-state \((\mu = \mu P)\) and entropy rate.

<table>
<thead>
<tr>
<th>Size</th>
<th>(P)</th>
<th>(H(p))</th>
<th>(H(\mathcal{X}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(P = \begin{pmatrix} 1-p &amp; p \ p &amp; 1-p \end{pmatrix})</td>
<td>(H(p))</td>
<td>(H(\mathcal{X}))</td>
</tr>
<tr>
<td>(N)</td>
<td>(P_{ij} = (1 - \bar{g}(\mathbf{1}(i)))^{n-1(j)} \bar{g}(\mathbf{1}(i))^{1(j)})</td>
<td>(n \sum_{i=1}^{N+1} \bar{\mu}_i H(\bar{g}(i - 1)))</td>
<td>(H(\mathcal{X}) - \sum_i \bar{\mu}_i^{N+1} \log (i \choose N))</td>
</tr>
</tbody>
</table>
Some Simulations
Minority Effects

\[ g(i) = \max(p, \min(\frac{i}{n}, 1 - p)) \]

\[ g(i) = p + (1 - 2p)\frac{i}{n} \]
**Entropy Rates**

**Omni-Network: Observe all vs. observe portion**

![Graphs showing entropy rates for different observations]

**Extension to \( N = 5 \) grid**

![Graph showing entropy for a 5x5 grid]
Conformist Grids

- Like a Markov Chain in space-time: \( H(X|X^-) = H(X|X^n) \)
- Assume a torus (no edge effects)
- Equilibrium Properties
- Non-Equilibrium Properties
- Conformist Grid Channels
Equilibrium Properties

- Distribution within \((X, X^n) \sim \mu_{N+1}\)
- Entropy of neighborhood, \(H(X_n|X) = H(\mu_{N+1}) - 1\)
- Entropy of neighbor, \(H(X_1|X) = H(\mu_{N+1}) - H(\mu_{N+1}|X, X_1) - 1\)
- Adjacent mutual information,
  \[
  I(X_1, X) = 2 - H(\mu_{N+1}) + H(\mu_{N+1}|X, X_1)
  \]
- Entropy from time evolution: \(H(Y|X) = I((\mu_{N+1}|X)P; Y)\)
Entropy from time evolution: \( H(Y|X = x, X^n = x^n) = H(g(x, x^n)) \)

\[
P(Y = y|X^n = x^n) = \sum_{x \in \{0, 1\}} p(x) P(Y = y|X = x, X^n = x^n)
\]

\[
P(Y = 1|X^n = x^n) = \frac{1}{2} [g(0, x^n) + g(1, x^n)]
\]

\[
P(Y = 0|X^n = x^n) = \frac{1}{2} [1 - g(0, x^n) + 1 - g(1, x^n)]
\]

\[
H(Y|X^n = x^n) = - \sum_{y = \{0, 1\}} p(y|x^n) \log p(y|x^n)
\]
Conformist Grid Channels

A simple channel, a relay channel, an interference channel, a 2-way (feedback) channel, a multiple access channel.
Consider a receiver surrounded by transmitters. According to network information theory, the limiting equation is

$$\sum_{i=1}^{N} R_i \leq I(X_1, \ldots, X_N; Y)$$

The information is maximized at

$$I(X_n; Y) = 1 - H\left(\frac{p + q}{2}\right)$$

with $p = \bar{g}(0)$ and $q = \bar{g}(1)$. 
Inverse Neighbor Channel

Neighbor channel information is upper bound between a node and its neighbor.

For a sender and receiver $K$ grid cells away,

- $P_1 = 1 - \frac{p+q}{2}$
- $P_K = 1 - P_{K-1} \frac{p+q}{2}$
- $P = \prod_{i=1}^{K} P_i$
- $I(Z_K; X) = 1 - H(P)$