Entropy rates of non-deterministic cellular automata

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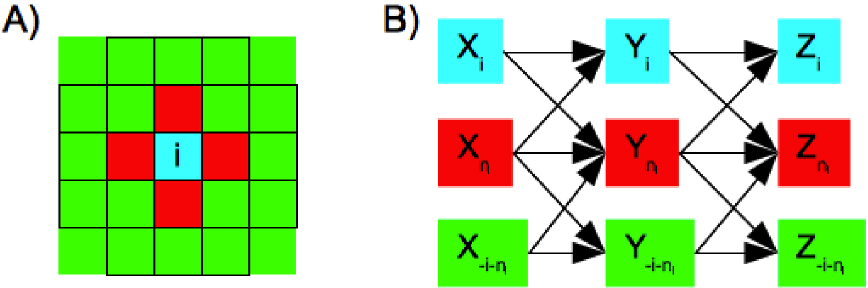
Introduction

- Understanding how information flows through spatially distributed media.
- Studying the "conformist" grid: how do people make decisions?

Basic Setup

- Cellular automata (CA), on a grid, in discrete time
- Non-deterministic state machines
- At each time, each node observes its neighbors, and draws from a Bernoulli distribution, $B(p)$.
- $0 < p < 1$, p increases with the number of neighbors of value 1, and is symmetric (if all values are flipped, $p' = 1 - p$).

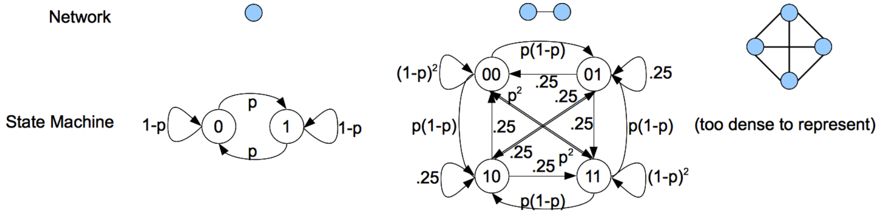
Notation



Notation

X, X_0:	value of given node at given time
X^n	collection of neighbors of node X
X_i $(i > 0)$	the value of neighbor of X
X_n	the number of neighbors of X with value 1
X^{-i}	collection of values excluding value i
Y:	value of given node at next time

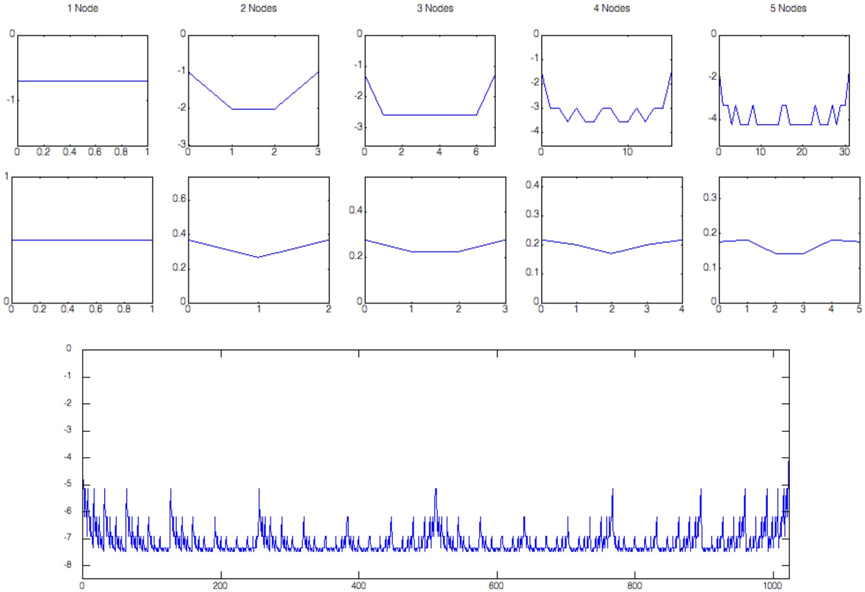
Omni-Conformist Network



Markov Chains Determine steady-state ($\mu = \mu P$) and entropy rate.

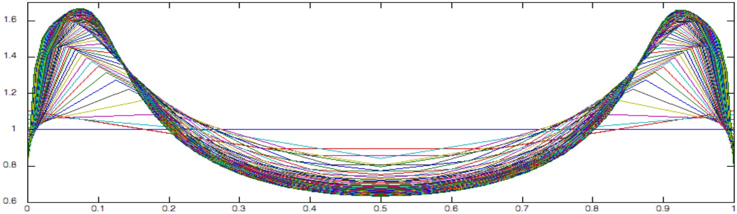
Size	P	$\mathcal{H}(\mathcal{X})$
1	$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$	$H(p)$
N	$P_{ij} = (1 - \bar{g}(\mathbf{1}(i)))^{n-\mathbf{1}(j)} \bar{g}(\mathbf{1}(i))^{\mathbf{1}(j)}$	$n \sum_{i=1}^{N+1} \bar{\mu}_i H(\bar{g}(i-1))$
N	$P_{ij} = \binom{N}{j-1} (1 - g(i))^{N-j+1} g(i)^{j-1}$	$\mathcal{H}(\mathcal{X}) - \sum_i \bar{\mu}_{i=1}^{N+1} \log \binom{N}{i-1}$

Some Simulations

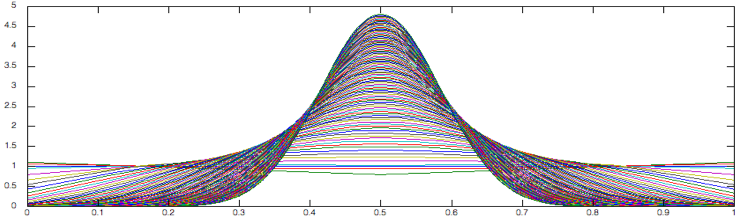


Minority Effects

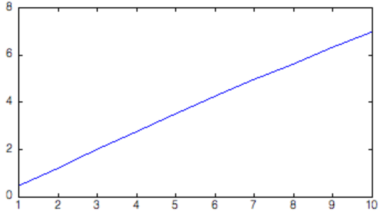
$$g(i) = max(p, min(\frac{i}{n}, 1 - p))$$

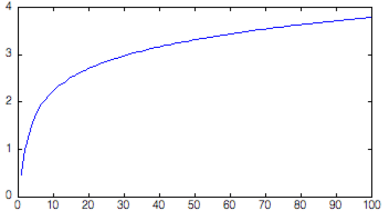


$$g(i) = p + (1 - 2p)\frac{i}{n}$$

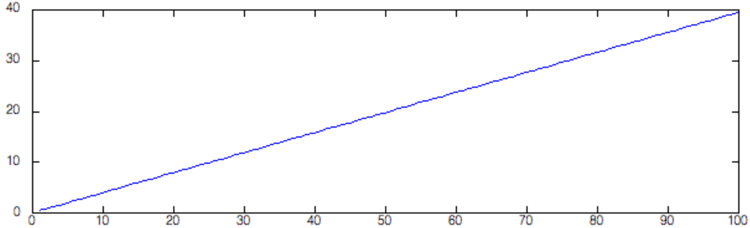


Omni-Network: Observe all vs. observe portion





Extension to $N = 5$ grid



Conformist Grids

- Like a Markov Chain in space-time: $H(X|X^-) = H(X|X^n)$
- Assume a torus (no edge effects)

- Equilibrium Properties
- Non-Equilibrium Properties
- Conformist Grid Channels

Equilibrium Properties

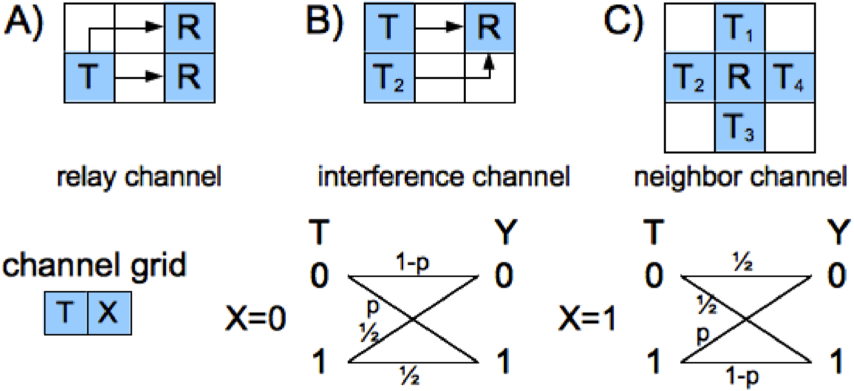
- Distribution within $(X, X^n) \sim \mu_{N+1}$
- Entropy of neighborhood, $H(X_n|X) = H(\mu_{N+1}) - 1$
- Entropy of neighbor, $H(X_1|X) = H(\mu_{N+1}) - H(\mu_{N+1}|X, X_1) - 1$
- Adjacent mutual information, $I(X_1, X) = 2 - H(\mu_{N+1}) + H(\mu_{N+1}|X, X_1)$
- Entropy from time evolution: $H(Y|X) = I((\mu_{N+1}|X)P; Y)$

Non-Equilibrium Properties

- Entropy from time evolution: $H(Y|X = x, X^n = x^n) = H(g(x, x^n))$
- $P(Y = y|X^n = x^n) = \sum_{x \in \{0,1\}} p(x) P(Y = y|X = x, X^n = x^n)$
- $P(Y = 1|X^n = x^n) = \frac{1}{2}[g(0, x^n) + g(1, x^n)]$
- $P(Y = 0|X^n = x^n) = \frac{1}{2}[1 - g(0, x^n) + 1 - g(1, x^n)]$
- $H(Y|X^n = x^n) = -\sum_{y=\{0,1\}} p(y|x^n) \log p(y|x^n)$

Conformist Grid Channels

A simple channel, a relay channel, an interference channel, a 2-way (feedback) channel, a multiple access channel.



Neighbor Channel

Consider a reciever surrounded by transmitters.

According to network information theory, the limiting equation is

$$\sum_{i=1}^{N} R_i \leq I(X_1, \ldots, X_N; Y)$$

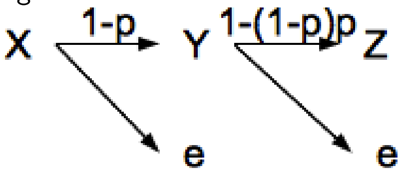
The information is maximized at

$$I(X_n; Y) = 1 - H(\frac{p+q}{2})$$

with $p = \bar{g}(0)$ and $q = \bar{g}(1)$.

Inverse Neighbor Channel

Neighbor channel information is upper bound between a node and its neighbor.



For a sender and receiver K grid cells away,

- $P_1 = 1 - \frac{p+q}{2}$
- $P_K = 1 - P_{K-1}\frac{p+q}{2}$
- $P = \prod_{i=1}^{K} P_i$
- $I(Z_K; X) = 1 - H(P)$